3.1

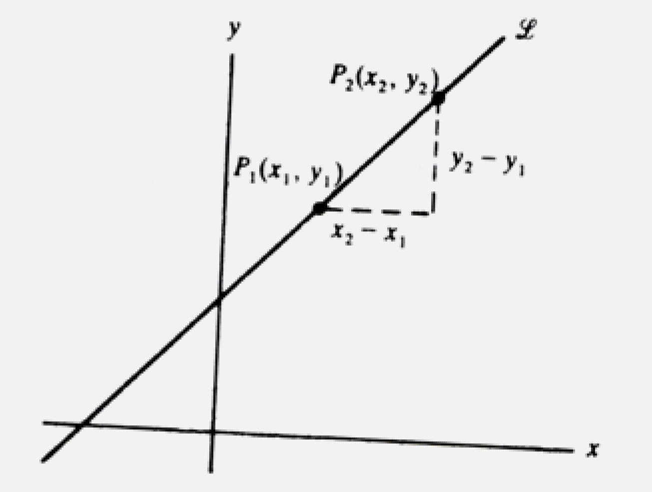
Slope of a Line

Learning objectives:

- To define slope of a line
- To derive the relationship between the slopes of parallel and perpendicular lines
- To derive a formula for the angle between two given lines in terms of their slopes
  And
- To solve related problems.

Slope of a Line

Given two points $P_1(x_1, y_1)$ *and* $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ *and* $\Delta y = y_2 - y_1$ the ***run*** and the ***rise***, respectively, between P_1 and P_2. Two such points always determine a unique straight line passing through them. We call the line P_1P_2.



The constant

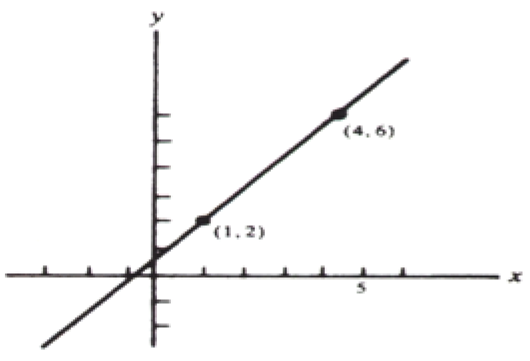
$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the slope of the *non-vertical line P_1P_2*.

Example 1:

The slope of the line joining the points (1,2) and (4,6) is

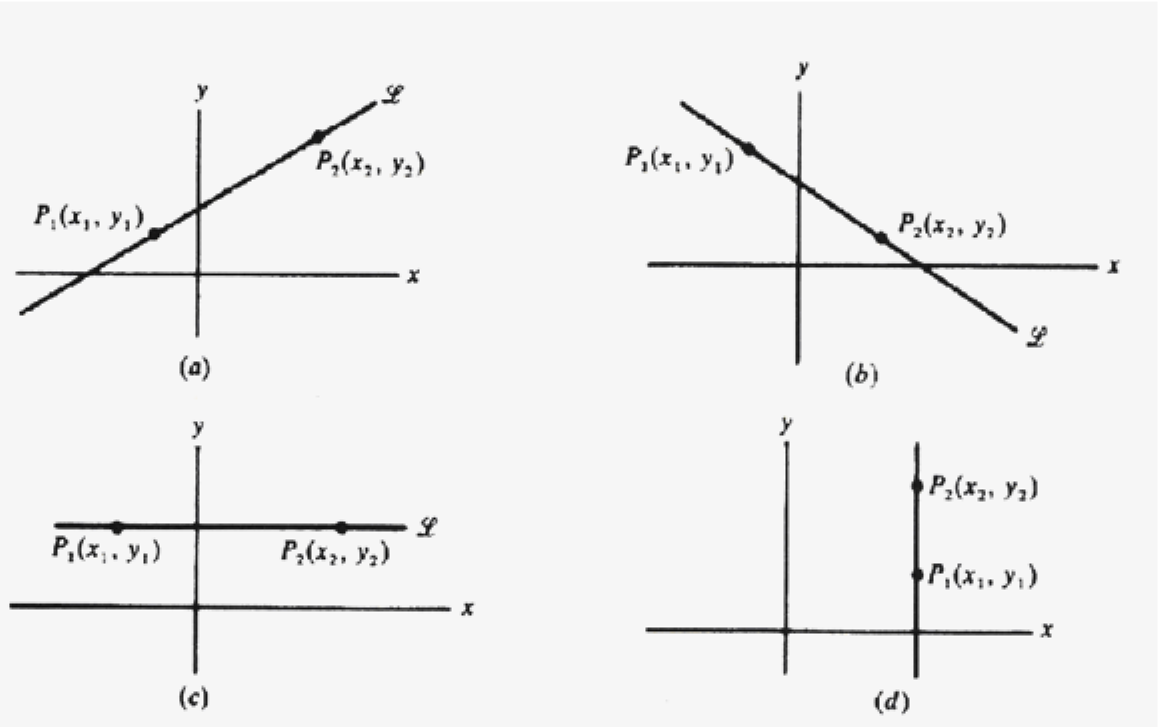
$$m = \frac{6-2}{4-1} = \frac{4}{3}$$



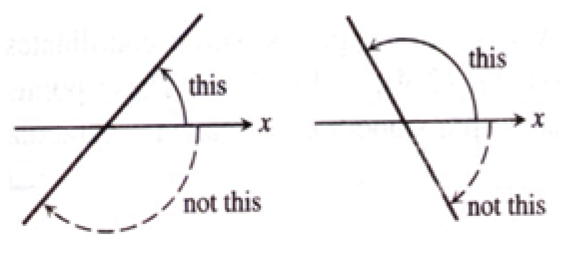
The slope tells us the direction (uphill, downhill) and steepness of a line. A line with positive slope rises uphill to the right and the line with negative slope falls downhill to the right. The greater the absolute value of the slope, the more rapid the rise or fall.

The slope of a horizontal line is zero.

The slope of a vertical line is undefined. Since the run Δx is zero for a vertical line, we cannot form the ratio m.



The direction and steepness of a line can also be measured with an angle. *The angle of inclination of a line that crosses the x-axis is the smallest counterclockwise angle from the x-axis to the line.*

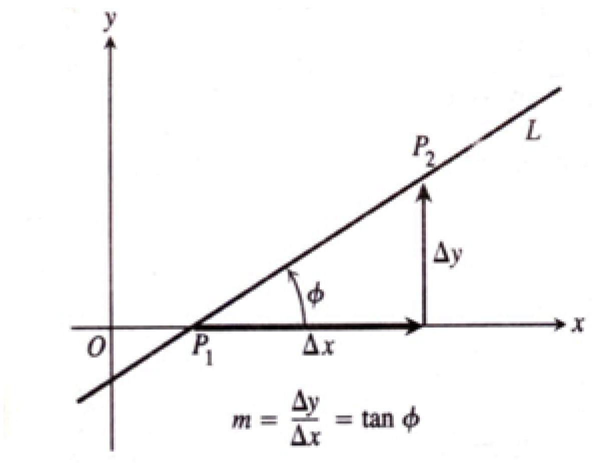


The inclination of a horizontal line is 0°.

The inclination of a vertical line is 90°.

If φ is the inclination of a line, then $0 \le \varphi < 180°$.

The relationship between the slope m of a non-vertical line and the line's inclination is $m = tan\ \varphi$



Example 2:

The slope of the line joining the points (1, 0) and (5, 4) is given by

$$m = \frac{4-0}{5-1} = \frac{4}{4} = 1$$

The angle of inclination of the line is given by

$$\varphi = \arctan m = \tan^{-1} 1 = \frac{\pi}{4}$$

3.2

Various Forms of Equation of a Line

Learning objectives:

- To derive the equation of a line in various forms
- To derive the equation of a line in
  Point-slope form
  Slope-intercept form
  Two point form
  Intercept form
  And Normal form
- To derive the parametric equations of a line
  And
- To solve related problems.

Equation of a Straight Line

Suppose r is the radius of the circle and C its circumference. Both r and C are variables. The Circumference and the radius of a circle are related by the equation $C = 2\pi r$. For any given radius, there is one and only one circumference. So we say r is an independent variable and C is a dependent variable. We can also view that when C changes, r changes according to the relation $r = \frac{C}{2\pi}$. So, here we can say that C is the independent variable and r is the dependent variable.

Thus, if there is a relation between two variables x and y, then if one is regarded as the independent variable, then other will be the dependent variable.

The equation of a line is an expression showing a relation between the x-coordinate and the y-coordinate of all the points that lie on the straight line. The relation is called a function with x as independent variable and y as dependent variable.

Lines determined by Point and Slope

We can write an equation for a non-vertical straight line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$. *If* $P(x, y)$ is any other point on L, then

$$\frac{y - y_1}{x - x_1} = m$$

$$y - y_1 = m(x - x_1)$$

The equation

$$y = y_1 + m(x - x_1)$$

is the **point-slope equation** of the line that passes through the point (x_1, y_1) and has slope m.

Example 1: Write an equation for the line through the point $(2, 3)$ with slope $-\frac{3}{2}$.

Solution:

$$y = 3 - \frac{3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + 6$$

Example 2: Write an equation for the line through $(-2, -1)$ and $(3, 4)$.

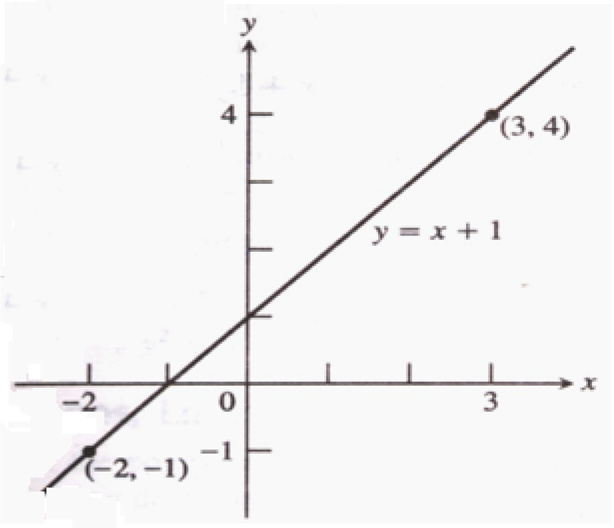
Solution:

$$m = \frac{-1 - 4}{-2 - 3} = \frac{-5}{-5} = 1$$

We can use this slope with either of the two given points. With the point $(x_1, y_1) = (-2, -1)$,

$$y = -1 + 1.(x - (-2))$$

$$y = -1 + x + 2$$



From the preceding description of the line we see that one point on the straight line and the direction of the straight line will determine it. Thus the fixation of a straight line requires the specification of two quantities. For example, we may specify slope and intercept or two points lying on the straight line and likewise. This will lead to various forms of equations which are known as the ***standard forms*** for the equation of straight line.

3.3

Parallel and Perpendicular Lines

Learning objectives:

- To derive conditions for two lines $\boldsymbol{a_1x + b_1y + c_1 = 0}$ and $\boldsymbol{a_2x + b_2y + c_2 = 0}$ to be $(\boldsymbol{i})$ parallel and $(\boldsymbol{ii})$perpendicular.
  And
- To practice problems on parallel and perpendicular lines.

Parallel Lines

Lines that are parallel have equal angles of inclination, and therefore they have the same slope .

Two straight lines whose equations are given in terms of their slopes are parallel when their slopes are same, or, in other words, if their equations differ only in the constant term. Therefore, the straight line $ax + by + c' = 0$ is parallel to the straight line $ax + by + c = 0$.

Again the equation $a(x - x') + b(y - y') = 0$ clearly represents a straight line which passes through the point(x', y') and is parallel to the line $ax + by + c = 0$.

The condition for the two lines $\boldsymbol{a_1x + b_1y + c_1 = 0}$ and $\boldsymbol{a_2x + b_2y + c_2 = 0}$ to be parallel

The condition is obtained by equating their slopes

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

That is, $a_1b_2 - a_2b_1 = 0$ is the condition for the given lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ to be parallel

If two equations $\boldsymbol{ax + by + c = 0}$ and $\boldsymbol{a'x + b'y + c' = 0}$ represent the same straight line, then

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

This can be proved as follows:

As the two equations represent the same straight line, they will have the same slope and the same y intercept. Thus

$$-\frac{a}{b} = -\frac{a'}{b'} \text{ and } -\frac{c}{b} = -\frac{c'}{b'}$$

Therefore

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

Example 1:

Find the equation to the straight line, which passes through the point $(4, -5)$, and which is parallel to the straight line $3x + 4y + 5 = 0$

Solution:

Any straight line which is parallel to the given line is of the form

$$3x + 4y + k = 0$$

The straight line passes through $(4, -5)$. Therefore,

$$3 \times 4 + 4 \times (-5) + k = 0$$

$$k = 8$$

Thus, the equation of the required line is

$$3x + 4y + 8 = 0$$

3.4

General Equation of a Line

Learning objectives:

- To study the general equation of a line.
- To study the reductions of the general equation of line to standard forms.
- To introduce the concept of positive and negative sides of a line and to derive a condition for a given two points to lie on the same side or opposite side of a line.
- To derive the equation of line passing through a given point and making a given angle with a given line.
  And
- To practice related problems.

The General equation of a Line

We have seen that all the forms of the equations of straight lines are only of the first degree in x and y. An equation that is expressible in the form

$$Ax + By + C = 0$$

where $A, B,$ and C are constants and at least one of A ,B is not zero, is called a *first-degree equation* in x and y. It is called the *general linear equation* in x and y since its graph always represents a line.
Every first degree equation of the form $Ax + By + C = 0$, where at least one of A, B is not zero, is called the ***general equation*** of a straight line.
Such a general equation can be reduced to the standard forms of the equations of lines.
We demonstrate the reduction of the general linear equation to the standard forms in the following.

Lines Parallel to Coordinate axes

Suppose $B = 0$, then A is not zero. Then the equation becomes $x = -\frac{C}{A}$,which is the equation of a straight line parallel to the $y - axis$, at $-\frac{C}{A}$ units from it.
On the other hand, if $A = 0$; then B is not zero and the equation becomes $y = -\frac{C}{B}$,which is the equation of a straight line parallel to the $x - axis$, at $-\frac{C}{B}$ units from it.

Slope Intercept Form

If $B \neq 0$, then we divide by B and solve for y, getting

$$y = -\frac{A}{B}x - \frac{C}{B}$$

This is the ***slope-intercept form*** of the equation of the line, with slope $-\frac{A}{B}$ and y −intercept $-\frac{C}{B}$.

Example 1: Find the slope and y −intercept of the line $8x + 5y = 20$.

Solution:

$$5y = -8x + 20 \quad \Rightarrow \quad y = -\frac{8}{5}x + 4$$

The slope $m = -\frac{8}{5}$ and the y −intercept is $c = 4$

3.5

Distance of a Point from a Line

Learning objectives:

- To derive a formula for the distance of a point from a line.
- To find the distance between two parallel lines.

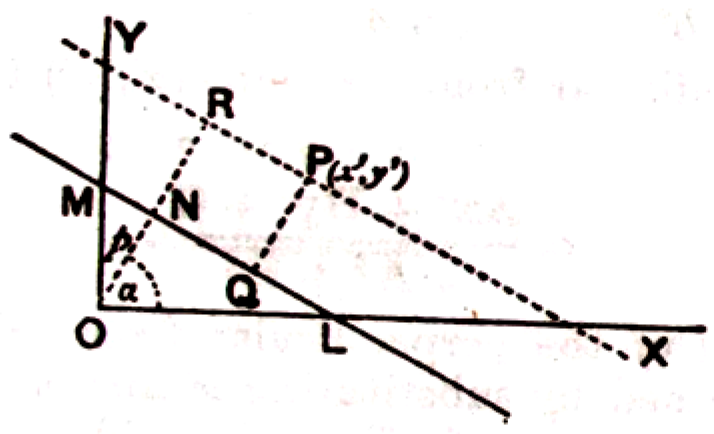
  And
- To practice the related problems.

Distance of a Point from a Line

Let the equation of the given straight line LM be

$$x\cos\alpha + y\sin\alpha - p = 0$$

So, in the figure, $ON = p$ and angle $XON = \alpha$



Let the given point P be (x', y'). Through P draw PR parallel to the given line. PQ is the required distance.

If OR is p', the equation to PR is

$$x\cos\alpha + y\sin\alpha - p' = 0.$$

Since this passes through the point (x', y'), we have

$$x'\cos\alpha + y'\sin\alpha - p' = 0.$$

So, $\quad p' = x'\cos\alpha + y'\sin\alpha$

∴ The Required distance $= p' - p = x'\cos\alpha + y'\sin\alpha - p$

The length of the required perpendicular is therefore obtained by substituting x' and y' for x and y in the given equation.

Since the normal form of an equation is equivalent to the general form, the following result follows.

The length of the perpendicular from (x', y') on $ax + by + c = 0$, where c *is negative*, is obtained by substituting x' and y' for x and y in the given equation and dividing the result so obtained by the square root of the sum of the squares of the coefficients of x and y.

Required distance $= \frac{ax' + by' + c}{\sqrt{a^2+b^2}}$

It follows that the length of the perpendicular from the origin is $\frac{c}{\sqrt{a^2+b^2}}$.

In practice we take, the length of the perpendicular from a point $P(x', y')$ to the straight line $ax + by + c = 0$ ------(1) as

$$\frac{|ax' + by' + c|}{\sqrt{a^2+b^2}}$$

and from this it follows that, the length of perpendicular from the origin to the line (1) is $\frac{|c|}{\sqrt{a^2+b^2}}$

Example 1: Find the distance of the point $(-3,4)$ from the line $3x + 4y - 5 = 0$.

Solution:

Distance $= \frac{3(-3)+4(-4)-5}{\sqrt{a^2+b^2}} = \frac{|3(-3) + 4(4) - 5|}{\sqrt{3^2 + 4^2}} = \frac{2}{5}$

Example 2: Find the points on the *y*-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3.

Solution:

Let $(0, k)$ is the required point.

$$\frac{|4(0) - 3k - 12|}{\sqrt{4^2 + (-3)^2}} = 3$$

$$\Rightarrow \quad \frac{-3k - 12}{5} = \pm 3$$

$$\therefore \quad -3k - 12 = 15 \quad \text{and} \quad -3k - 12 = -15$$

$$\Rightarrow \quad k = -9 \quad \text{and} \quad k = 1$$

The required points are $(0,1)$ and $(0, -9)$

3.6

Family of Lines

Learning objectives:

- To derive a formula for the coordinates of the point of intersection of two lines.
- To derive the condition for the concurrency of three lines.
- To derive the equation for the family of lines through the intersection of two given lines.

  And
- To practice related problems.

Intersection of Straight Lines

Consider two lines whose equations are

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

The coordinates of the point of intersection satisfy both the equations. Solving these equations,

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad , \quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Therefore, the lines *intersect* at the point with the above coordinates provided $a_1b_2 - a_2b_1 \neq 0$. This means that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

In case $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the lines are ***parallel*** and ***there is no point of intersection***. On the other hand if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the lines are ***coincident*** and every point on them is a point of intersection.

Example 1:

Find the point of intersection of the line $\frac{x}{3} - \frac{y}{4} = 0, \frac{x}{2} + \frac{y}{3} = 1$.

Solution:

The given lines are $4x - 3y = 0$, $3x + 2y - 6 = 0$

Solving these equations, we get

$$x = \frac{(-3)\times(-6) - (2)\times 0}{4\times 2 - 3\times(-3)} = \frac{18}{17} \qquad , \qquad y = \frac{0\times 3 - (-6)\times 4}{4\times 2 - 3\times(-3)} = \frac{24}{17}$$